COUNTABLY-NORMED SPACES, THEIR DUAL, AND THE GAUSSIAN MEASURE

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Abstract. Here we present an overview of countably-normed spaces. We discuss the main topologies—weak, strong, and inductive—placed on the dual of a countably-normed space and discuss the σ–fields generated by these topologies. In particular, we show that under certain conditions the strong and inductive topologies coincide and the σ–fields generated by the weak, strong, and inductive topologies are equal. With this σ–field, we develop a Gaussian measure on the dual of a nuclear space. The purpose in mind is to provide the background material for many of the results used is White Noise Analysis.

1. Topological Vector Spaces

In this section we review the basic notions of topological vector spaces along and provide proofs a few useful results.

1.1. **Topological Preliminaries.** Let E be a real vector space.

A *vector topology* τ on E is a topology such that addition $E \times E \to E : (x, y) \mapsto x + y$ and scalar multiplication $\mathbb{R} \times E \to E : (t, x) \mapsto tx$ are continuous. If E is a complex vector space we require that $\mathbb{C} \times E \to E : (\alpha, x) \mapsto \alpha x$ be continuous.

It is useful to observe that when E is equipped with a vector topology, the translation maps

$$t_x : E \to E : y \mapsto y + x$$

are continuous, for every $x \in E$, and are hence also homeomorphisms since $t_x^{-1} = t_{-x}$.

A *topological vector space* is a vector space equipped with a vector topology.

Recall that a *local base* of a vector topology τ is a family of open sets $\{U_\alpha\}_{\alpha \in I}$ containing 0 such that if W is any open set containing 0 then W contains some U_α. A set W that contains an open set containing x is called a *neighborhood* of x. If U is any open set and x any point in U then $U - x$ is an open neighborhood of 0 and hence contains some U_α, and so U itself contains a neighborhood $x + U_\alpha$ of x:

(1.1) *If U is open and $x \in U$ then $x + U_\alpha \subset U$, for some $\alpha \in I$.*

Doing this for each point x of U, we see that each open set is the union of translates of the local base sets U_α.

If $\mathcal{U}_x$ denotes the set of all neighborhoods of a point x in a topological space X, then $\mathcal{U}_x$ has the following properties:

(1) $x \in U$ for all $U \in \mathcal{U}_x$.
(2) if $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_x$, then $U \cap V \in \mathcal{U}_x$.
(3) if $U \in \mathcal{U}_x$ and $U \subset V$, then $V \in \mathcal{U}_x$.

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(4) if $U \in \mathcal{U}_x$, then there is some $V \in \mathcal{U}_x$ with $U \in \mathcal{U}_y$ for all $y \in V$. (taking V to be the interior of U is sufficient).

Conversely if X is any set and a non-empty collection of subsets $\mathcal{U}_x$ is given for each $x \in X$, then when the conditions above are satisfied by the $\mathcal{U}_x$, exactly one topology can be defined on X in such a way to make $\mathcal{U}_x$ the set of neighborhoods of x for each $x \in X$. A set $V \subset X$ is called *open* if for each $x \in V$, there is a $U \in \mathcal{U}_x$ with $U \subset V$. [6]

In most cases of interest a topological vector space has a local base consisting of convex sets. We call such spaces *locally convex topological vector spaces*.

In a topological vector space there is the notion of bounded sets. A set D in a topological vector space is said to be *bounded*, if for every neighborhood U of 0 there is some $\lambda > 0$ such that $D \subset \lambda U$. If $\{U_\alpha\}_{\alpha \in I}$ is a local base, then it is easily seen that D is bounded if and only if to each U_α there corresponds $\lambda_\alpha > 0$ with $D \subset \lambda_\alpha U_\alpha$. [6]

A set A in a vector space E is said to be *absorbing* if given any $x \in E$ there is an η such that $x \in lA$ for all $|l| \geq \eta$. The set A is called *balanced* if, for all $x \in A$, $lx \in A$ whenever $|l| \leq 1$. Also, a set A in a vector space E is call *symmetric* if $-A = A$. Finally, although the next concept is very common, the term we use for it is not, so we make a formal definition:

Definition 1.1. A subset A of a topological space X is *limit point compact* if every infinite subset of A has a limit point.

Remark 1.2. The term *limit point compact* is not the standard term for spaces with the above property. In fact, I do not believe there is a standard term. I have seen it called "Fréchet compactness", "relative sequential compactness", and the "Bolzano-Weierstrass property". The term *limit point compact* was taken directly from Munkres [5]. It is my personal favorite term; at the very least it is descriptive.

1.2. **Bases in Topological Vector Spaces.** Here we take the time to prove some general, but very useful, results about local bases for topological vector spaces. Most of the results in this subsection are taken from Robertson [6].

Lemma 1.3. *Every topological vector space E has a base of balanced neighborhoods.*

Proof. Let U be a neighborhood of 0 in E. Consider the function $h : \mathbb{C} \times E \to E$ given by $h(l, x) = lx$. Since E is a topological vector space, h is continuous at $l = 0, x = 0$. So there is a neighborhood V and $\epsilon > 0$ with $lx \in U$ for $|l| \leq \epsilon$ and $x \in V$. Hence $lV \subset U$ for $|l| \leq \epsilon$. Therefore $\frac{\epsilon}{\alpha} V \subset U$ for all α with $|\alpha| \geq 1$. Thus $\epsilon V \subset U' = \bigcap_{|\alpha| \geq 1} \alpha U \subset U$. Now since V is a neighborhood of 0 so is ϵV. Hence U' is a neighborhood of 0. If $x \in U'$ and $0 \leq |l| \leq 1$, then for $|\alpha| \geq 1$, we have $x \in \frac{\alpha}{l} U$ (since $\left|\frac{\alpha}{l}\right| \geq 1$). So $lx \in \alpha U$ for $|l| \leq 1$. Hence $lx \in U'$. Therefore U' is balanced. □

Lemma 1.4. *Let E be a vector space. Let $\mathcal{B}$ be a collection of subsets of E satisfying:*

(i) *if $U, V \in \mathcal{B}$, then there exist $W \in \mathcal{B}$ with $W \subset U \cap V$.*
(ii) *if $U \in \mathcal{B}$ and $l \neq 0$, then $lU \in \mathcal{B}$.*
(iii) *if $U \in \mathcal{B}$, then U is balanced, convex, and absorbing.*

Then there is a topology making E a locally convex topological vector space with $\mathcal{B}$ the base of neighborhoods of 0.

Proof. Let $\mathcal{A}$ be the set of all subsets of E that contain a set of $\mathcal{B}$. For each x take $x+\mathcal{A}$ to be the set of neighborhoods of x. We need to see that (1)-(4) are satisfied from subsection 1.1.

For (1), we have to show $x \in A$ for all $A \in x+\mathcal{A}$. Note that since each $U \in \mathcal{B}$ is absorbent, there exists a non-zero l such that $0 \in lU$. But then $0 \in l^{-1}lU = U$. So each $U \in \mathcal{B}$ contains 0. So $x \in A$ for all $A \in x+\mathcal{A}$.

For (2), we have to show that if $A, B \in \mathcal{A}$, then $(x+A)\cap(x+B) \in x+\mathcal{A}$ for each $x \in E$. Recall that $U \subset A$ and $V \subset B$ for some $U, V \in \mathcal{B}$. So $U \cap V \subset A \cap B$. By the first hypothesis, there is a $W \in \mathcal{B}$ with $W \subset U \cap V \subset A \cap B$. Thus $A \cap B \in \mathcal{A}$ and hence $(x+A)\cap(x+B) \in x+\mathcal{A}$ for each $x \in E$.

Next (3) is clear from the definition of $\mathcal{A}$, since if $A \in \mathcal{A}$ and $A \subset B$ then $B \in \mathcal{A}$.

Finally for (4), we must show that if $x+A \in x+\mathcal{A}$, then there is an $V \in x+\mathcal{A}$ with $x+A \in y+\mathcal{A}$ for all $y \in V$. If $A \in \mathcal{A}$ take a $U \in \mathcal{B}$ with $U \subset A$. Now we see that $x+A$ is a neighborhood of each point $y \in x+\frac{1}{2}U$. Since $y \in x+\frac{1}{2}U$ we have $y-x \in \frac{1}{2}U$. Thus $y-x+\frac{1}{2}U \subset \frac{1}{2}U+\frac{1}{2}U \subset A$. Hence $y+\frac{1}{2}U \subset x+A$. Thus $x-y+A \supset \frac{1}{2}U$. So $x-y+A \in \mathcal{A}$. Therefore $y+x-y+A = x+A \in y+\mathcal{A}$.

To prove continuity of addition, let $U \in \mathcal{B}$. Then if $x \in a+\frac{1}{2}U$ and $y \in b+\frac{1}{2}U$, we have $x+y \in a+b+U$.

Finally, to see that scalar multiplication, lx, is continuous at $x=a, l=\alpha$, we should find δ_1 and δ_2 such that $lx-\alpha a \in U$ whenever $|l-\alpha| < \delta_1$ and $x \in a+\delta_2 U$. Since U is absorbing, there is a η with $a \in \eta U$. Take δ_1 so that $0 < \delta_1 < \frac{1}{2\eta}$ and take δ_2 so that $0 < \delta_2 < \frac{1}{2(|\alpha|+\delta_1)}$. Now observe

$$\begin{aligned} lx-\alpha a &= l(x-a)+(l-\alpha)a \\ &\in (|\alpha|+\delta_1)\delta_2 U + \delta_1 \eta U \\ &\subset \frac{1}{2}U+\frac{1}{2}U \subset U. \end{aligned}$$

Thus we are done. □

1.3. **Topologies generated by families of topologies.** Let $\{\tau_\alpha\}_{\alpha\in I}$ be a collection of topologies on a space. It is natural and useful to consider the least upper bound topology τ, i.e. the coarsest topology containing all sets of $\cup_{\alpha\in I}\tau_\alpha$. In our setting, we work with each τ_α a vector topology on a vector space E.

Theorem 1.5. *The least upper bound topology τ of a collection $\{\tau_\alpha\}_{\alpha\in I}$ of vector topologies is again a vector topology. If $\{W_{\alpha,i}\}_{i\in I_\alpha}$ is a local base for τ_α then a local base for τ is obtained by taking all finite intersections of the form $W_{\alpha_1,i_1}\cap\cdots\cap W_{\alpha_n,i_n}$.*

Proof. Let $\mathcal{B}$ be the collection of all sets which are of the form $W_{\alpha_1,i_1}\cap\cdots\cap W_{\alpha_n,i_n}$.

Let τ' be the collection of all sets which are unions of translates of sets in $\mathcal{B}$ (including the empty union). Our first objective is to show that τ' is a topology on E. It is clear that τ' is closed under unions and contains the empty set. We have to show that the intersection of two sets in τ' is in τ'. To this end, it will suffice to prove the following:

$$\begin{gathered} \text{If } C_1 \text{ and } C_2 \text{ are sets in } \mathcal{B}, \text{ and } x \text{ is a point in} \\ \text{the intersection of the translates } a+C_1 \text{ and } b+C_2, \\ \text{then } x+C \subset (a+C_1)\cap(b+C_2) \text{ for some } C \text{ in } \mathcal{B}. \end{gathered} \tag{1.2}$$

Clearly, it suffices to consider finitely many topologies τ_α. Thus, consider vector topologies $\tau_1, ..., \tau_n$ on E.

Let $\mathcal{B}_n$ be the collection of all sets of the form $B_1 \cap \cdots \cap B_n$ with B_i in a local base for τ_i, for each $i \in \{1, ..., n\}$. We can check that if $D, D' \in \mathcal{B}_n$ then there is an $G \in \mathcal{B}_n$ with $G \subset D \cap D'$.

Working with B_i drawn from a given local base for τ_i, let z be a point in the intersection $B_1 \cap \cdots \cap B_n$. Then there exist sets B_i', with each B_i' being in the local base for τ_i, such that $z + B_i' \subset B_i$ (this follows from our earlier observation (1.1)). Consequently,

$$z + \cap_{i=1}^n B_i' \subset \cap_{i=1}^n B_i.$$

Now consider sets C_1 and C_2, both in $\mathcal{B}_n$. Consider $a, b \in E$ and suppose $x \in (a+C_1) \cap (b+C_2)$. Then since $x-a \in C_1$ there is a set $C_1' \in \mathcal{B}_n$ with $x-a+C_1' \subset C_1$; similarly, there is a $C_2' \in \mathcal{B}_n$ with $x - b + C_2' \subset C_2$. So $x + C_1' \subset a + C_1$ and $x + C_2' \subset b + C_2$. So

$$x + C \subset (a + C_1) \cap (b + C_2),$$

where $C \in \mathcal{B}_n$ satisfies $C \subset C_1 \cap C_2$.

This establishes (1.2), and shows that the intersection of two sets in τ' is in τ'.

Thus τ' is a topology. The definition of τ' makes it clear that τ' contains each τ_α. Furthermore, if any topology σ contains each τ_α then all the sets of τ' are also open relative to σ. Thus

$$\tau' = \tau,$$

the topology generated by the topologies τ_α.

Observe that we have shown that if $W \in \tau$ contains 0 then $W \supset B$ for some $B \in \mathcal{B}$.

Next we have to show that τ is a vector topology. The definition of τ shows that τ is translation invariant, i.e. translations are homeomorphisms. So, for addition, it will suffice to show that addition $E \times E \to E : (x, y) \mapsto x + y$ is continuous at $(0, 0)$. Let $W \in \tau$ contain 0. Then there is a $B \in \mathcal{B}$ with $0 \in B \subset W$. Suppose $B = B_1 \cap \cdots \cap B_n$, where each B_i is in the given local base for τ_i. Since τ_i is a vector topology, there are open sets $D_i, D_i' \in \tau_i$, both containing 0, with

$$D_i + D_i' \subset B_i.$$

Then choose C_i, C_i' in the local base for τ_i with $C_i \subset D_i$ and $C_i' \subset D_i'$. Then

$$C_i + C_i' \subset B_i.$$

Now let $C = C_1 \cap \cdots \cap C_n$, and $C' = C_1' \cap \cdots \cap C_n'$. Then $C, C' \in \mathcal{B}$ and $C + C' \subset B$. Thus, addition is continuous at $(0, 0)$.

Now consider the multiplication map $\mathbb{R} \times E \to E : (t, x) \mapsto tx$. Let $(s, y), (t, x) \in \mathbb{R} \times E$. Then

$$sy - tx = (s - t)x + t(y - x) + (s - t)(y - x).$$

Suppose $F \in \tau$ contains tx. Then

$$F \supset tx + W',$$

for some $W' \in \mathcal{B}$. Using continuity of the addition map

$$E \times E \times E \to E : (a, b, c) \mapsto a + b + c$$

at $(0,0,0)$, we can choose $W_1, W_2, W_3 \in \mathcal{B}$ with $W_1 + W_2 + W_3 \subset W'$. Then we can choose $W \in \mathcal{B}$, such that

$$W \subset W_1 \cap W_2 \cap W_3$$

Then $W \in \mathcal{B}$ and

$$W + W + W \subset W'.$$

Suppose $W = B_1 \cap \cdots \cap B_n$, where each B_i is in the given local base for the vector topology τ_i. Then for s close enough to t, we have $(s-t)x \in B_i$ for each i, and hence $(s-t)x \in W$. Similarly, if y is τ–close enough to x then $t(y-x) \in W$. Lastly, if $s-t$ is close enough to 0 and y is close enough to x then $(s-t)(y-x) \in W$. So $sy - tx \in W'$, and so $sy \in F$, when s is close enough to t and y is τ–close enough to x. □

The above result makes it clear that if each τ_α has a convex local base then so does τ. Note also that if at least one τ_α is Hausdorff then so is τ.

A family of topologies $\{\tau_\alpha\}_{\alpha \in I}$ is *directed* if for any $\alpha, \beta \in I$ there is a $\gamma \in I$ such that

$$\tau_\alpha \cup \tau_\beta \subset \tau_\gamma.$$

In this case every open neighborhood of 0 in the generated topology contains an open neighborhood in one of the topologies τ_γ.

2. Countably–Normed Spaces

We begin with the basic definition of a *countably-normed space* and a *countably-Hilbert space.*

Definition 2.1. Let V be a topological vector space over $\mathbb{C}$ with topology given by a family of norms $\{|\cdot|_n; n = 1, 2, \ldots\}$. Then V is a *countably-normed space.* The space V is called a *countably-Hilbert space* if each $|\cdot|_n$ is an inner product norm and V is complete with respect to its topology.

Remark 2.2. By considering the new norms $\|v\|_n = (\sum_{k=1}^{n} |v|_k{}^2)^{\frac{1}{2}}$ we may assume that the family of norms $\{|\cdot|_n; n = 1, 2, \ldots\}$ is increasing, i.e.

$$|v|_1 \le |v|_2 \le \cdots \le |v|_n \le \cdots, \forall v \in V$$

If V is a countably-normed space, we denote the completion of V in the norm $|\cdot|_n$ by V_n. Then V_n is by definition a Banach space. Also in light of Remark 2.2 we can assume that

$$V \subset \cdots \subset V_{n+1} \subset V_n \subset \cdots \subset V_1$$

Lemma 2.3. *The inclusion map from V_{n+1} into V_n is continuous.*

Proof. Consider an open neighborhood of 0 in V_n given by

$$B_n(0, \epsilon) = \{v \in V_n; |v|_n < \epsilon\}$$

Let $i_{n+1,n} : V_{n+1} \to V_n$ be the inclusion map. Now

$$i_{n+1,n}^{-1}(B_n(0,\epsilon)) = \{v \in V_{n+1}; |v|_n < \epsilon\} \supset B_{n+1}(0,\epsilon) \text{ since } |v|_n \le |v|_{n+1}$$

Therefore $i_{n+1,n}$ is continuous. □

Proposition 2.4. *Let V be a countably-normed space. Then V is complete if and only if $V = \bigcap_{n=1}^{\infty} V_n$.*

Proof. Suppose $V = \bigcap_{n=1}^{\infty} V_n$ and $\{v_k\}_{k=1}^{\infty}$ is Cauchy in V. By definition $\{v_k\}_{k=1}^{\infty}$ is Cauchy in V_n for all n. Since V_n is complete, a limit $v^{(n)}$ exist in V_n. Using that the inclusion map $i_{n+1,n} : V_{n+1} \to V_n$ is continuous (by Lemma 2.3) and that

$$V \subset \cdots \subset V_{n+1} \subset V_n \subset \cdots \subset V_1$$

we have that all the $v^{(n)}$ are the same and belong to each V_n. Thus they are in $V = \bigcap_{n=1}^{\infty} V_n$. Let us call this element $v \in V$.

Since $|v_k - v^{(n)}|_m \to 0$ for all m we have that $|v_k - v|_m \to 0$ for all m. Hence $v = \lim_{k\to\infty} v_k$ in V. Thus V is complete.

Conversely, let V be complete and take $v \in \bigcap_{n=1}^{\infty} V_n$. We need to show v is in V. For each n we can find $v_n \in V$ such that $|v - v_n|_n < \frac{1}{n}$ (using that V is dense in V_n). Now for any $k < n$ we have $|v - v_n|_k \le |v - v_n|_n < \frac{1}{n}$. Thus $\lim_{n\to\infty} |v - v_n|_k = 0$. This gives us that $\{v_n\}$ is Cauchy with respect to all norms $|\cdot|_k$ where $k = 1, 2, \ldots$

Let $\overline{v} = \lim_{n\to\infty} v_n$ in V. Since for all k we have $\overline{v}, v \in V_k$ and $\lim_{n\to\infty} |\overline{v} - v_n|_k = 0$, we see that $v = \overline{v}$. Thus $v \in V$ and we have $V \supset \bigcap_{n=1}^{\infty} V_n$. That $V \subset \bigcap_{n=1}^{\infty} V_n$ is obvious, since $V \subset V_n$ for all n. □

2.1. **Open Sets in V.** In light of Theorem 1.5, we see that a local base for V is given by sets of the form:

$$B = B_{n_1}(\epsilon_1) \cap B_{n_2}(\epsilon_2) \cap \cdots \cap B_{n_k}(\epsilon_k)\,,$$

where $B_{n_i}(\epsilon_i) = \{v \in V; |v|_{n_i} < \epsilon_i\}$ is the $|\cdot|_{n_i}$–ball of radius ϵ_i in V.

Proposition 2.5. *Let V be a countably-normed space. For every element B of the local base for V there exist n and $\epsilon > 0$ such that $B_n(\epsilon) \subset B$.*

Proof. Let $B = B_{n_1}(\epsilon_1) \cap B_{n_2}(\epsilon_2) \cap \cdots \cap B_{n_k}(\epsilon_k)$ be an element of the local base for V. Then take $n = \max_{1\le j\le k} n_j$ and $\epsilon = \min_{1\le j\le k} \epsilon_j$. Observe $B_n(\epsilon) \subset B$ since for $v \in B_n(\epsilon)$ we have $|v|_{n_j} \le |v|_n < \epsilon \le \epsilon_j$ for any $j \in \{1, 2, \ldots, k\}$. Thus $v \in B$. □

Corollary 2.6. *Let V be a countably-normed space. Then a local base for V is given by the collection $\{B_n(\frac{1}{k})\}_{n,k=1}^{\infty}$.*

Corollary 2.7. *Let V be a countably-normed space. Then a local base for V is given by the collection $\{B_k(\frac{1}{k})\}_{k=1}^{\infty}$. Moreover we have that $B_1(1) \supset B_2(\frac{1}{2}) \supset \cdots$*

Proof. Let U be a neighborhood of 0. By Corollary 2.6 there are positive integers n and k such that $B_n(\frac{1}{k}) \subset U$. If $n \ge k$, we have that $B_n(\frac{1}{n}) \subset B_n(\frac{1}{k})$ since $\frac{1}{n} \le \frac{1}{k}$. If $n \le k$, then $B_k(\frac{1}{k}) \subset B_n(\frac{1}{k})$ since $|v|_k < \frac{1}{k}$ gives us that $|v|_n \le |v|_k < \frac{1}{k}$.

For $m \ge k$ we have that $B_m(\frac{1}{m}) \subset B_k(\frac{1}{k})$ since $|v|_k \le |v|_m$ and $\frac{1}{m} < \frac{1}{k}$. □

2.2. **Bounded Sets in V.** Recall that a subset D of a countably-normed space V is said to be *bounded* if for any neighborhood U of zero in V there is a positive number λ such that $D \subset \lambda U$ (see subsection 1.1). This leads us to the following useful proposition:

Proposition 2.8. *A set D in a countably-normed space V is bounded if and only if $\sup_{v\in D} |v|_n < \infty$ for all $n \in \{1, 2, \ldots\}$.*

Proof. ($\Rightarrow$) Suppose D is a bounded set in V. Take the open neighborhood $B_n(1) = \{v \in V; |v|_n < 1\}$ in V. Since D is bounded in V there is an $l > 0$ such that $D \subset lB_n(1)$. Thus $\sup_{v\in D} |v|_n \le l$.

($\Leftarrow$) Suppose U is a neighborhood of 0 in V. Then by Proposition 2.5 there is an $B_n(\epsilon) \subset U$. Let $\sup_{v\in D} |v|_n = M < \infty$. Then $D \subset \frac{M+1}{\epsilon} B_n(\epsilon) \subset \frac{M+1}{\epsilon} U$. So D is bounded. $\square$

2.3. **The Dual.** Again take V to be a countably-normed space associated with an increasing sequence of norms $\{|\cdot|_n\}_{n=1}^{\infty}$ and let V_n be the completion of V with respect to the norm $|\cdot|_n$. We denote the dual space of V by V'. Let $\langle \cdot, \cdot \rangle$ denote the bilinear pairing of V' and V.

Of course, each Banach space V_n also has a dual, which we denote by V_n'. We use the notation to $|\cdot|_{-n}$ to denote the operator norm on the Banach space V_n'. The relationship between V' and each V_n' is discussed in the next proposition.

Proposition 2.9. *The dual of a countably-normed space V is given by $V' = \bigcup_{n=1}^{\infty} V_n'$ and we have the inclusions*

$$V_1' \subset \cdots \subset V_n' \subset V_{n+1}' \subset \cdots V'$$

Moreover, for $f \in V_n'$ we have $|f|_{-n} \geq |f|_{-n-1}$.

Proof. ($\supset$) Take $v' \in V_n'$. Then v' is continuous on V_n with topology coming from the norm $|\cdot|_n$. Thus v' is continuous on V, since $V \subset V_n$ and the norm $|\cdot|_n$ is one of the norms generating the topology on V.

($\subset$) Take $v' \in V'$. Since v' is continuous on V the set

$$v'^{-1}(-1, 1) = \{v \in V \,;\, |\langle v', v\rangle| < 1\}$$

is open in V. So we can find a member B of the local base for V such that $B \subset v'^{-1}(-1, 1)$. By to Proposition 2.5 we have that $B_n(\epsilon) \subset v'^{-1}(-1, 1)$ for some positive integer n and some $\epsilon > 0$.

Thus for all $v \in V$ with $|v|_n < \epsilon$ we have that $|\langle v', v\rangle| < 1$. Since V is dense in V_n, if $v \in V_n$ and $|v|_n \leq \epsilon$ then $|\langle v', v\rangle| \leq 1$. Thus $v' \in V_n'$.

To see that $V_n' \subset V_{n+1}'$ take $f \in V_n'$. Then for all $v \in V_n$ we have that

$$|f(v)| \leq |f|_{-n}|v|_n \leq |f|_{-n}|v|_{n+1}$$

Since $V_{n+1} \subset V_n$, the above holds for all $v \in V_{n+1}$. Thus $f \in V_{n+1}'$ and $|f|_{-n-1} \leq |f|_{-n}$. $\square$

Proposition 2.10. *A linear functional f on V is continuous if and only if f is bounded on bounded sets of V.*

Proof. ($\Rightarrow$) Let f be a continuous linear functional on V. Then f is in V'. So $f = \langle v', \cdot\rangle$ for some $v' \in V'$. Now by Proposition 2.9, $v' \in V_n'$ for some n. Let $D \subset V$ be bounded. By Proposition 2.8 we have that $\sup_{v\in D} |v|_n = M < \infty$. Using this we see that $\sup_{v\in D} |\langle v', v\rangle| \leq M|v'|_{-n} < \infty$. Thus $f = \langle v', \cdot\rangle$ is bounded on bounded sets.

($\Leftarrow$) Suppose f is bounded on bounded sets. Consider the local base sets $B_1(1) \supset B_2\left(\frac{1}{2}\right) \supset \cdots$ in V as in Corollary 2.7. By contradiction we assume that f is not in V'. Then f is not in V_k' for any k. So f is not continuous on V_k and hence not bounded on $B_k\left(\frac{1}{k}\right)$. Hence we can find a v_k in $B_k\left(\frac{1}{k}\right)$ such that $|f(v_k)| > k$. The sequence $\{v_k\}_{k=1}^{\infty}$ goes to 0 in V. Thus $\{v_k\}_{k=1}^{\infty}$ must be bounded. But then by hypothesis, $\{f(v_k)\}_{k=1}^{\infty}$ should be bounded. But by construction it is not, a contradiction. $\square$

Corollary 2.11. *A linear functional f on V is continuous if and only if f is bounded on some neighborhood of* 0 *in* V.

Proof. Suppose f is bounded on some neighborhood U of 0 in V. Then for any $\alpha > 0$, f is bounded on αU. Let D be a bounded set in V. Then $D \subset lU$ for some $l > 0$. So f is bounded on D and hence continuous by Proposition 2.10 $\square$

There are several topologies one can put on the dual space V'. The three most common are the weak, strong, and inductive topologies. In the following sections we discuss the properties of these three topologies and compare them against one another. Throughout this discussion, the topology on V_n' is taken to be the usual strong topology (i.e. the topology induced by the operator norm on V_n' as the dual of the Banach space V_n).

2.4. **Bounded Sets of V revisited.** Let V be a countably-normed space. With the notion of the dual V' of V behind us (see subsection 2.3), we can formulate a better understanding of bounded sets in V. We begin with the following simple definition:

Definition 2.12. A set $D \subset V$ is said to be *weakly bounded* if given a set $N(v'; \epsilon) = \{v \in V; |\langle v', v\rangle| < \epsilon\}$ there is a $l > 0$ such that $D \subset lN(v'; \epsilon)$.

Theorem 2.13. *Suppose V is a countably-normed space with dual V'. Let $D \subset V$. Then the following are equivalent:*

(1) *D is bounded.*
(2) *D is weakly bounded.*
(3) *The values of each $v' \in V'$ are bounded on D.*
(4) *For all n, we have* $\sup_{v\in D} |v|_n < \infty$.

Proof. We have already shown that (1) and (4) are equivalent in Proposition 2.8.

((1) $\Rightarrow$ (2)) Suppose D is bounded in V. Take a $v' \in V'$. Then $v' \in V_n'$ for some n. For $v \in D$ we have $|\langle v', v\rangle| \leq |v'|_{-n}|v|_n \leq |v'|_{-n}M_n$ where $M_n = \sup_{v\in D} |v|_n$. Thus we have $D \subset \frac{2|v'|_{-n}M_n}{\epsilon} N(v'; \epsilon)$. So D is weakly bounded.

((2) $\Rightarrow$ (3)) Suppose D is weakly bounded in V. Take $v' \in V'$. By assumption $D \subset lN(v'; \epsilon)$ for some $l > 0$. So for $v \in D$ we have $|\langle v', v\rangle| \leq l\epsilon$.

((3) $\Rightarrow$ (4)) Consider $D \subset V \subset V_n$. By hypothesis all $v' \in V'$ are bounded on D. In particular all $v' \in V_n' \subset V'$ are bounded on D. This means the linear functionals $\{\langle \cdot, v\rangle; v \in D\}$ are pointwise bounded on V_n'. Thus we can apply the uniform boundedness principle to see that $\sup_{v\in D} |v|_n < \infty$. $\square$

2.5. **The Metric on V.** Let V be a countably-normed space. Define the function $\rho : V \times V \to [0, \infty)$ by

$$\rho(v, u) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|v - u|_n}{1 + |v - u|_n}. \tag{2.1}$$

First observe that ρ is a metric on V. From the above definition it is obvious that $\rho(v, v) = 0$ and $\rho(v, u) > 0$ for all $u \neq v$. It is also clear that $\rho(v, u) = \rho(u, v)$. We have left to check the triangle inequality. To verify the triangle inequality it is sufficient to show that

$$\frac{|v + u|_n}{1 + |v + u|_n} \leq \frac{|v|_n}{1 + |v|_n} + \frac{|u|_n}{1 + |u|_n}.$$

To show this, we first note that the function $f : [0, \infty) \to [0, 1)$ given by $f(t) = \frac{t}{1+t}$ is increasing. Thus

$$\begin{aligned} \frac{|v+u|_n}{1+|v+u|_n} &\le \frac{|v|_n + |u|_n}{1+|v|_n+|u|_n} \\ &= \frac{|v|_n}{1+|v|_n+|u|_n} + \frac{|u|_n}{1+|v|_n+|u|_n} \\ &\le \frac{|v|_n}{1+|v|_n} + \frac{|u|_n}{1+|u|_n}. \end{aligned}$$

Proposition 2.14. *The metric ρ on V has the following properties:*

(1) $\rho(v, u) = \rho(v - u, 0)$
(2) *If $v_k \to 0$ in V, then $\rho(v_k, 0) \to 0$.*

Proof. That $\rho(v, u) = \rho(v - u, 0)$ for all $u, v \in V$ is obvious from the definition.

For (2), let $v_k \to 0$ in V. Then $\lim_{k\to\infty} |v_k|_n \to 0$ for each n. So, for a given $\epsilon > 0$, take N so that $\frac{1}{2^N} < \frac{\epsilon}{2}$. Take K such that for any $k > K$ we have $|v_k|_n < \frac{\epsilon}{2}$ for all $1 \le n \le N$. Then for $k > K$ we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|v_k|_n}{1+|v_k|_n} = \sum_{n=1}^{N} \frac{1}{2^n} \frac{|v_k|_n}{1+|v_k|_n} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{|v_k|_n}{1+|v_k|_n} < \frac{\epsilon}{2} + \frac{1}{2^N} < \epsilon.$$

Therefore $\rho(v_k, 0) \to 0$ as $k \to \infty$. □

As you may have guessed, we would not take the time to talk about this metric unless it proved useful in some way. Well, it turns out that the topology induced by this metric is identical to the original topology on V.

Theorem 2.15. *The topology on the countably-normed space V induced by the metric ρ is equivalent to the original topology on V (i.e. the topology induced by the family of norms $\{|\cdot|_n\}_{n=1}^{\infty}$).*

Proof. Applying Proposition 2.14, it is sufficient to consider the neighborhoods $\{v \in V\,;\, |v|_n < \delta\}$ of 0 in V and the sets $\{v \in V\,;\, \rho(v, 0) < \epsilon\}$ for $\epsilon, \delta > 0$ and $n \in \{1, 2, \ldots\}$. We have to show that every $\{v \in V\,;\, |v|_n < \delta\}$ contains some $\{v \in V\,;\, \rho(v, 0) < \epsilon\}$ and conversely.

Consider a neighborhood $\{v \in V\,;\, |v|_n < \delta\}$ in V. If $v \in V$ satisfies $\rho(v, 0) < \epsilon$, then $\frac{1}{2^n} \frac{|v|_n}{1+|v|_n} < \epsilon$ and thus

$$|v|_n < \frac{2^n \epsilon}{1 - 2^n \epsilon} = \frac{2^n}{\frac{1}{\epsilon} - 2^n}.$$

So, take $\epsilon > 0$ such that

$$0 < \frac{2^n}{\frac{1}{\epsilon} - 2^n} < \delta,$$

and we have $\{v \in V\,;\, \rho(v, 0) < \epsilon\} \subset \{v \in V\,;\, |v|_n < \delta\}$.

Now consider a set $\{v \in V\,;\, \rho(v, 0) < \epsilon\}$. Assume, by contradiction, there is no n and $\delta > 0$ such that $\{v \in V\,;\, |v|_n < \delta\} \subset \{v \in V\,;\, \rho(v, 0) < \epsilon\}$. Then for each k we can find $v_k \in \left\{v \in V\,;\, |v|_k < \frac{1}{k}\right\}$ such that v_k is not in $\{v \in V\,;\, \rho(v, 0) < \epsilon\}$. This gives us a sequence $\{v_k\}_{k=1}^{\infty}$ that tends to 0 in V but not with respect to the metric ρ. This contradicts Proposition 2.14. □

From this it follows that V is a complete countably-normed space if and only (V, ρ) is a complete metric space. The following is a result which proves useful in a few theorems to come:

Lemma 2.16. *Given a closed convex symmetric absorbing set C in a complete countably-normed space V we can find a neighborhood U of 0 contained in C.*

Proof. Since C is absorbing we have that $V \subset \bigcup_{n=1}^{\infty} nC$. Knowing that V is a complete metric space we can apply the Baire category theorem to see that the closed set C is not nowhere dense. Thus the interior of C, C°, is not empty. Take v in C° and let U be a symmetric open set around 0 such that $v + U \subset C^{\circ}$ (e.g. take U to be one of the $B_k(\frac{1}{k})$ described in Corollary 2.7).

Because C is symmetric we have that $-v-U = -v+U$ is in C. Since C is convex it contains the convex hull of $v + U$ and $-v + U$. But this convex hull contains U; observe for any $w \in U$ we have that

$$w = \frac{(v + w) + (-v + w)}{2}.$$

Thus we are done. □

3. Weak Topology

The weak topology is the simplest topology placed on the dual of a countably-normed space. It is defined as follows:

Definition 3.1. The *weak topology* on the dual V' of a countably-normed space V is the coarsest vector topology on V' such that the functional $\langle \cdot, v\rangle$ is continuous for any $v \in V$.

In the following propositions, we prove some commonly used properties of the weak topology.

Proposition 3.2. *The weak topology on V' has a local base of neighborhoods given by sets of the form:*

$$N(v_1, v_2, \ldots, v_k; \epsilon) = \{v' \in V';\ |\langle v', v_j\rangle| < \epsilon,\ 1 \leq j \leq k\}.$$

Proof. In order for $\langle \cdot, v\rangle$ to be continuous for all $v \in V$ we need $\langle \cdot, v\rangle$ to be continuous at 0. Or equivalently, we require that $\langle \cdot, v\rangle^{-1}(-\epsilon, \epsilon) = N(v; \epsilon)$ be open for each $\epsilon \in \mathbb{R}$. Hence for each $v \in V$ we form the topology τ_v on V given by the local base $\{N(v; \epsilon)\}_{\epsilon>0}$. The weak topology is the least upper bound topology for the family $\{\tau_v\}_{v\in V}$ (see subsection 1.3). Thus, by Theorem 1.5, a local base for the weak topology is given by sets of the form

$$N(v_1, v_2, \ldots, v_k; \epsilon) = N(v_1; \epsilon) \cap N(v_2; \epsilon) \cap \cdots \cap N(v_k; \epsilon),$$

where $v_1, v_2, \ldots, v_k \in V$. □

Proposition 3.3. *The inclusion map $i'_n : V'_n \to V'$ is continuous when V' is given the weak topology.*

Proof. Consider the weak base neighborhood $N(v_1 \ldots v_k; \epsilon)$ where $v \in V$. Observe that

$$i_n'^{-1}\left(N(v_1 \ldots v_k; \epsilon)\right) = \{v' \in V'_n;\ |\langle v', v_j\rangle| < \epsilon,\ 1 \leq j \leq k\}.$$

Since for each j, $v_j \in V \subset V_n$ we have that the functional $\langle \cdot, v_j\rangle$ is continuous on V'_n. (Since V_n is a Banach space, $V_n \subset V''_n$.) Thus $\{v' \in V'_n;\ |\langle v', v_j\rangle| < \epsilon,\ 1 \leq j \leq k\}$, is open in V'_n, being the finite intersection of open sets. □

Proposition 3.4. *Let V be a countably-Hilbert space. Then the space V_n' is dense in V' when V' is endowed with the weak topology.*

Proof. Consider $v_0' \in V'$. An arbitrary neighborhood U of v_0' contains a set of the form $v_0' + N$ where $N = N(v_1, \dots, v_k; \epsilon) = \{v' \in V';\ |\langle v', v_j\rangle| < \epsilon,\ 1 \le j \le k\}$. We must find a $v_n' \in V_n'$ such that $v_n' \in v_0' + N$. That is $|\langle v_n' - v_0', v_j\rangle| < \epsilon$ for all $1 \le j \le k$.

Now $v_0' \in V_l'$ for some l since $V' = \bigcup_{n=1}^{\infty} V_n'$. If $l \le n$ we are done, since $V_l' \subset V_n'$ by Proposition 2.9. If $l > n$ a little more work needs to be done, but it is still very straightforward.

For clarity, we assume $k = 2$ and v_1, v_2 are independent unit vectors in V_n. (There is no harm in assuming this. We can just shrink ϵ suitably by dividing by the maximum of $|v_1|_n$ and $|v_2|_n$.) Suppose $\langle v_0', v_1\rangle = l_1$ and $\langle v_0', v_2\rangle = l_2$. Write v_2 as $v_2 = \alpha v_1 + \beta v_1^{\perp}$ where $v_1^{\perp}$ is a unit vector in the orthogonal complement of $\{v_1\}$ in V_n. Then $l_2 = \langle v_0', v_2\rangle = l_1\alpha + \beta\langle v_0', v_1^{\perp}\rangle$ or equivalently $\langle v_0', v_1^{\perp}\rangle = \frac{l_2 - l_1\alpha}{\beta}$. Consider $w = l_1 v_1 + \frac{l_2 - l_1\alpha}{\beta} v_1^{\perp}$. Now $w \in V_n$. Thus $\langle w, \cdot\rangle_n$ is in V_n', where $\langle \cdot, \cdot\rangle_n$ is the inner-product on V_n. We now observe that $\langle w, v_1\rangle_n = l_1$ and $\langle w, v_2\rangle_n = \langle w, \alpha v_1 + \beta v_1^{\perp}\rangle_n = l_1\alpha + l_2 - l_1\alpha = l_2$. Hence $\langle w, \cdot\rangle_n$ agrees with v_0' on v_1 and v_2. Therefore $w \in v_0' + N$ and we have that V_n' is dense in V'. □

4. Strong Topology

Recall the notion of bounded sets in a countably-normed space V (as in subsections 2.2 and 2.4). Using bounded sets in V we can define the *strong topology* on V'.

Definition 4.1. The *strong topology* on the dual V' of a countably-normed space V is defined to be the topology with a local base given by sets of the form

$$N(D; \epsilon) = \{v' \in V';\ \sup_{v \in D} |\langle v', v\rangle| < \epsilon\},$$

where D is any bounded subset of V and $\epsilon > 0$.

Taking D to be a finite set such as $\{v_1, v_2, \dots, v_k\}$, it is clear that the strong topology is finer than the weak topology.

Proposition 4.2. *The inclusion map $i_n' : V_n' \to V'$ is continuous when V' is given the strong topology.*

Proof. Consider the neighborhood $N(D; \epsilon) = \{v' \in V';\ \sup_{v \in D} |\langle v', v\rangle| < \epsilon\}$ where D is a bounded set in V and $\epsilon > 0$. Now

$$i_n'^{-1}(N(D; \epsilon)) = \Big\{v' \in V_n';\ \sup_{v \in D} |\langle v', v\rangle| < \epsilon\Big\}.$$

Let $\sup_{v \in D} |v|_n = M$. Take v_0' in $i_n'^{-1}(N(D; \epsilon))$ and let $c_0 = \sup_{v \in D} |\langle v_0', v\rangle| < \epsilon$. Consider the open set $B(v_0', \frac{\epsilon - c_0}{M+1}) = \{v' \in V_n';\ |v' - v_0'|_{-n} < \frac{\epsilon - c_o}{M+1}\}$. We assert that $B(v_0', \frac{\epsilon - c_0}{M+1}) \subset i_n'^{-1}(N(D; \epsilon))$.

Take $v' \in B(v_0', \frac{\epsilon - c_0}{M+1})$. Then $|v' - v_0'|_{-n} < \frac{\epsilon - c_0}{M+1}$. This gives us the following

$$\sup_{v \in D} \big|\langle v' - v_0', \tfrac{v}{|v|_n}\rangle\big| < \frac{\epsilon - c_0}{M+1}.$$

Thus $\sup_{v \in D} |\langle v' - v_0', v\rangle| < \epsilon - c_0$, since $|v|_n \le M$ when $v \in D$. From this we see that $\sup_{v \in D} |\langle v', v\rangle| < \epsilon$. Therefore $v' \in i_n'^{-1}(N(D; \epsilon))$. □

4.1. **Strongly bounded sets of V'.** When V' is endowed with the strong topology, a bounded set $B \subset V'$ is called *strongly bounded*. (Likewise when V' has the weak topology, B is said to be *weakly bounded*). Strongly bounded sets have many nice properties, which we will prove in this section. First let us begin with the following definition:

Definition 4.3. A set $B \subset V'$ is said to be *bounded on the set $A \subset V$* if

$$\sup_{v' \in B, v \in A} |\langle v', v\rangle| < \infty.$$

Lemma 4.4. *A set $B \subset V'$ is strongly bounded if and only if it is bounded on each bounded set $D \subset V$.*

Proof. ($\Rightarrow$) Let $B \subset V'$ be strongly bounded and let D be a bounded set of V. Consider the neighborhood of V' given by

$$N(D;1) = \Big\{ v' \in V';\ \sup_{v \in D} |\langle v', v\rangle| < 1 \Big\}.$$

Since B is bounded there exists an $l > 0$ such that $B \subset lN(D;1)$ or equivalently $\frac{1}{l}B \subset N(D;1)$. Then for $v' \in B$ we have $\frac{v'}{l} \in N(D;1)$. Thus $|\langle v, v'\rangle| \leq l$ for any $v \in D$. Therefore B is bounded on the set D.

($\Leftarrow$) Suppose B is bounded on each bounded set $D \subset V$. Consider a neighborhood $N(D;\epsilon)$ of 0 in V'. By hypothesis, $\sup_{v' \in B, v \in D} |\langle v', v\rangle| = M < \infty$. So for any $v' \in B$ we have that $|\langle \frac{\epsilon v'}{M+1}, v\rangle| < \epsilon$ when $v \in D$. Thus $\frac{\epsilon}{M+1}B \subset N(D;\epsilon)$ or equivalently $B \subset \frac{M+1}{\epsilon}N(D;\epsilon)$. Hence B is bounded. □

Lemma 4.5. *A set $B \subset V'$ is strongly bounded if and only if there exists k such that B is bounded on $B_k(\frac{1}{k})$.*

Proof. ($\Rightarrow$) As per Corollary 2.7, consider the local base sets $B_1(1) \supset B_2(\frac{1}{2}) \supset \cdots$ of V. By contradiction suppose that B is not bounded on $B_k(\frac{1}{k})$ for any k. Then for every k there exist a $v_k \in B_k(\frac{1}{k})$ and a $v'_k \in B$ such that $|\langle v'_k, v_k\rangle| > k$. The sequence $\{v_k\}$ goes to 0, thus it must be bounded. So by Lemma 4.4 there must exist a positive number M such that $|\langle v', v_k\rangle| \leq M$ for all $v' \in B$ and all $k \in \{1, 2, \ldots\}$. This contradicts the way v'_k and v_k were chosen.

($\Leftarrow$) Conversely, let $B \subset V'$ be bounded on some $B_k(\frac{1}{k}) \subset V$. Take a bounded set D in V. Then $D \subset lB_k\big(\frac{1}{k}\big)$ for some $l > 0$. Thus B is bounded on D, since B is bounded on $lB_k\big(\frac{1}{k}\big)$. Thus by Lemma 4.4, B is bounded. □

Theorem 4.6. *A set $B \subset V'$ is strongly bounded if and only if $B \subset V'_k$ for some k and B is bounded in the norm $|\cdot|_{-k}$ of V'_k.*

Proof. ($\Leftarrow$) Let $B \subset V'_k$ be bounded in the norm $|\cdot|_{-k}$ by some $M > 0$ (i.e. $\sup_{v' \in B} |v'|_{-k} < M$). Consider the set $B_k(1) = \{v \in V; |v|_k < 1\}$. Then for $v' \in B$ and $v \in B_k(1)$ we have that $|\langle v', v\rangle| \leq M$. Thus B is bounded on $B_k(1)$ and hence on $B_k(\frac{1}{k})$. Therefore B is strongly bounded by Lemma 4.5.

($\Rightarrow$) Conversely suppose B is a strongly bounded set in V'. Then by Lemma 4.5 there is a k such that B is bounded on the set $B_k(\frac{1}{k}) = \{v \in V; |v|_k < \frac{1}{k}\}$. That is there is an $M < \infty$ such that $|\langle v', v\rangle| \leq M$ for all $v' \in B$ and all $v \in B_k\big(\frac{1}{k}\big)$.

Let $N_k \subset V_k$ be given by $N_k = \{v \in V_k; |v|_k < \frac{1}{k}\}$. Since V is dense in V_k we have that

$$\sup_{v' \in B, v \in N_k} |\langle v', v\rangle| \leq M.$$

From the above we see for any $v' \in B$ and unit vector $v \in V_k$ we have that $|\langle v', \frac{v}{(k+1)}\rangle| < M$. Hence $|v'|_{-k} \le (k+1)\, M$. Thus for any $v' \in B$ we have that $v' \in V_k'$ and $|v'|_{-k} \le (k+1)\, M$. □

4.2. **Reflexivity.** Just as we can discuss the dual V' of V, we can also talk about the dual of V'. Of course, this depends on the topology we put on V'. As we will see it turns out that $V'' = V$ as sets if V' is given the weak or strong topology (and V is a countably-Hilbert space). We can also put a topology on V''. We construct this topology from the strongly bounded sets in V'. For each set B in V' that is strongly bounded and each $\epsilon > 0$ form the neighborhood

$$N(B;\epsilon) = \Big\{\hat{v} \in V'';\ \sup_{v' \in B} |\langle \hat{v}, v'\rangle| < \epsilon\Big\}.$$

Take the collection of all sets $N(B;\epsilon)$ as our local base in V''. We call this topology the *strong topology on* V''. Given this topology we will also see that V'' is homeomorphic to V.

Proposition 4.7. *Let V be a countably-Hilbert space. Then $V = V''$ when V' is given the weak or strong topology.*

Proof. Consider $v \in V$ and the corresponding linear functional $\hat{v}$ on V' given by

$$\langle \hat{v}, v'\rangle = \langle v', v\rangle, \qquad \text{where } v' \in V'.$$

Observe that $\langle \hat{v}, \cdot\rangle$ is continuous since $\langle \hat{v}, \cdot\rangle^{-1}(-\epsilon, \epsilon) = \{v' \in V'; |\langle v', v\rangle| < \epsilon\}$ which is open in the weak (and hence the strong) topology on V'.

Also note that if $\hat{u} = \hat{v}$, then $\langle v', v\rangle = \langle v', u\rangle$ for all $v' \in V'$. Thus $v = u$. Therefore the correspondence $v \to \hat{v}$ is injective.

We now show that the correspondence $v \to \hat{v}$ is surjective. Take $v'' \in V''$. Then v'' is continuous on V'. Since, by Proposition 2.9, $V' = \bigcup_{n=1}^{\infty} V_n'$ we have that $v'' \in V_n''$ for all n. But $V_n = V_n''$ since V_n is a Hilbert space. Thus v'' can be considered as an element of V_n for all n. Since V is a countably-Hilbert space we have that $\cap_{n=1}^{\infty} V_n = V$ by Proposition 2.4. Thus $v'' \in V$ and we have that $v \to \hat{v}$ is surjective. □

Theorem 4.8. *If V is a countably-Hilbert space, then V'' is homeomorphic to V when V'' is given the strong topology.*

Proof. From Proposition 4.7 we already see that $V = V''$. We now need to see that the correspondences $\hat{v} \to v$ and $v \to \hat{v}$ are continuous.

First we consider the continuity of $v \to \hat{v}$. Let $N(B;\epsilon)$ be a neighborhood of 0 in V''. So we have that B is a strongly bounded set in V'. By Theorem 4.6 we know that $B \subset V_k'$ for some k and is bounded in the norm $|\cdot|_{-k}$. Let us call $\sup_{v' \in B} |v'|_{-k} = M < \infty$. Consider the neighborhood $B_k\big(\frac{\epsilon}{M}\big) \subset V$ given by $B_k\big(\frac{\epsilon}{M}\big) = \{v \in V; |v|_k < \frac{\epsilon}{M}\}$. Take a $v \in B_k\big(\frac{\epsilon}{M}\big)$. We need to see that $\hat{v} \in N(B;\epsilon)$. For any $v' \in B$ we have that

$$|\langle \hat{v}, v'\rangle| = |\langle v', v\rangle| \le |v'|_{-k} |v|_k < M \frac{\epsilon}{M} = \epsilon.$$

So $\hat{v} \in N(B;\epsilon)$. Thus $v \to \hat{v}$ is continuous.

Now consider $\hat{v} \to v$. Let $0 < \epsilon < 1$ and take $B_k(\epsilon) = \{v \in V; |v|_k < \epsilon\}$, a member of the local base for V (see subsection 2.1). Let $B \subset V'$ be given by

$$B = \{v' \in V'; |\langle v', v\rangle| \le 1, \text{ for all } v \in V_k \text{ with } |v|_k < \epsilon\}.$$

Note that B is strongly bounded by Theorem 4.6. So we can form the local base element $N(B;\epsilon)$ of V'' given by

$$N(B;\epsilon) = \left\{ \hat{v} \in V'';\ \sup_{v' \in B} |\langle \hat{v}, v' \rangle| < \epsilon \right\}.$$

Take a $\hat{v} \in N(B;\epsilon)$. Note that $\langle \frac{v}{|v|_k}, \cdot \rangle_k \in B$ since $\langle \frac{v}{|v|_k}, u \rangle_k \leq |u|_k$ for $u \in V_k$. Since $\hat{v} \in N(B;\epsilon)$ and $\langle \frac{v}{|v|_k}, \cdot \rangle_k \in B$, we must have $|\langle \frac{v}{|v|_k}, v \rangle_k| = |v|_k < \epsilon$. Therefore $v \in B_k(\epsilon)$. This proves the continuity of the map $\hat{v} \to v$. □

4.3. **Completeness in V'.** Suppose V' is given the strong topology. The convergence of a sequence of functionals $\{v'_k\}_{k=1}^{\infty}$ in V' to an element $v'_0 \in V'$ is called *strong convergence* and $\{v'_k\}_{k=1}^{\infty}$ is said to *converge strongly* to v'_0. Obviously $\{v'_k\}_{k=1}^{\infty}$ converging strongly to v'_0 is equivalent to $\{v'_k - v'_0\}_{k=1}^{\infty}$ converging strongly to 0. Thus a sequence $\{v'_k\}_{k=1}^{\infty}$ converges strongly to v'_0 if and only if for any bounded set $D \subset V$ and any number $\epsilon > 0$ there exists a $K > 0$ such that $v'_k - v'_0 \in N(D;\epsilon) = \{v' \in V';\ \sup_{v \in D} |\langle v', v \rangle| < \epsilon\}$ for all $k \geq K$. Hence a sequence $\{v'_k\}_{k=1}^{\infty}$ converges strongly to v'_0 if and only if $\{\langle v'_k, \cdot \rangle\}_{k=1}^{\infty}$ converges uniformly to $\langle v'_0, \cdot \rangle$ on each bounded set $D \subset V$. We say that a sequence $\{v'_k\}_{k=1}^{\infty}$ is *strongly Cauchy* (or *strongly fundamental*) if the sequence of numbers $\{\langle v'_k, v \rangle\}_{k=1}^{\infty}$ converges for each element $v \in V$ and the convergence is uniform on each bounded set $D \subset V$.

Theorem 4.9. *Let V be a countably-normed space. The dual V' of V is complete under the strong topology.*

Proof. Let $\{v'_k\}_{k=1}^{\infty}$ be a strongly Cauchy sequence in V'. Then for $v \in V$ we have that the sequence of numbers $\{\langle v'_k, v \rangle\}_{k=1}^{\infty}$ converges. We conveniently denote this limit by $\langle v', v \rangle$. For each $v \in V$ we have

$$\langle v', v \rangle = \lim_{k \to \infty} \langle v'_k, v \rangle.$$

This functional $\langle v', \cdot \rangle$ is clearly linear on V. We have to check that it is continuous. For this it is sufficient to see that $\langle v', \cdot \rangle$ is bounded on bounded sets (see Proposition 2.10). Let D be a bounded set in V. Observe that the functions $\{\langle v'_k, \cdot \rangle\}_{k=1}^{\infty}$ are bounded on D. Moreover they converge uniformly to $\langle v', \cdot \rangle$ on D. Hence there is a $K > 0$ such that $|\langle v' - v'_K, v \rangle| < 1$ for all v in D. Thus we have that

$$\sup_{v \in D} |\langle v', v \rangle| \leq \sup_{v \in D} |\langle v'_K, v \rangle| + 1 < \infty.$$

Therefore $\langle v', \cdot \rangle$ is bounded on bounded sets and hence continuous. So $v' \in V'$ and V' is complete with respect to the strong topology. □

4.4. **Comparing the Weak and Strong topology.** When a countably-normed space V is complete, many properties of the strong and weak topologies coincide. We will see that weakly and strongly bounded sets are one in the same. Also under suitable conditions, weak and strong convergence coincide.

Theorem 4.10. *Let V be a complete countably-normed space with dual V'. Every weakly bounded set in V' is strongly bounded.*

Proof. By Lemma 4.5 and Corollary 2.7 it is sufficient to show that a weakly bounded set B is bounded on some neighborhood of zero in V.

Let us define a set $C \subset V$ as follows:

$$C = \{v \in V; |\langle v', v\rangle| \leq 1 \text{ for all } v' \in B\} = \bigcap_{v' \in B} \{v \in V; |\langle v', v\rangle| \leq 1\}.$$

Observe that C is closed, being the intersection of closed sets, C is convex, being the intersection of convex sets, and C is symmetric, being the intersection of symmetric sets. Finally note that C is absorbent: Take $v \in V$. Since B is weakly bounded we must have $B \subset lN(v; 1)$ where $N(v; 1) = \{v' \in V'; |\langle v', v\rangle| < 1\}$ for some $l > 0$. Thus $|\langle v', v\rangle| \leq l$ for all $v' \in B$. Hence $\frac{v}{l} \in C$ or equivalently $v \in lC$.

So we can apply Lemma 2.16 to see that there is a neighborhood U of 0 in V such that $U \subset C$. Therefore the elements of B are uniformly bounded on U (by 1). Thus B is bounded on U and hence strongly bounded. □

Corollary 4.11. *Let V be a complete countably-normed space with dual V'. If a sequence $\{v'_k\}_{k=1}^{\infty}$ in V' converges pointwise (on each $v \in V$), then $\{v'_k\}_{k=1}^{\infty}$ is strongly bounded.*

Proof. Since $\{v'_k\}_{k=1}^{\infty}$ converges pointwise, it is weakly bounded. □

Corollary 4.12. *Let V be a complete countably-normed space with dual V'. Then V' is complete with respect to the weak topology.*

Proof. Take a Cauchy sequence $\{v'_k\}_{k=1}^{\infty} \subset V'$. Then by Corollary 4.11, we have that $\{v'_k\}_{k=1}^{\infty}$ is strongly bounded. Thus by Lemma 4.5, $\{v'_k\}_{k=1}^{\infty}$ is bounded on some neighborhood U of 0 in V. That is, there exists an $M > 0$ such that $|\langle v'_k, v\rangle| \leq M$ for all $v \in U$ and all $k \in \{1, 2, \ldots\}$.

Define v' by $\langle v', v\rangle = \lim_{k\to\infty}\langle v'_k, v\rangle$. Obviously, v' is linear. Observe that for all $v \in U$ we have

$$|\langle v', v\rangle| = \lim_{k\to\infty} |\langle v'_k, v\rangle| \leq M.$$

So v' is bounded on U and hence continuous (by Corollary 2.11). □

Of particular interest are countably-normed spaces such with the property that bounded sets are limit point compact (see Definition 1.1). These spaces have many wonderful properties, that do not hold in general for infinite-dimensional normed spaces. We make the following definition (the terminology comes from Gel'fand [1]):

Definition 4.13. A complete countably-normed space V in which all bounded sets are limit point compact is called *perfect*.

Remark 4.14. Since Theorem 2.15 gives us that V is metrizable, limit point compact can be replaced with compact or sequentially compact in the above definition. Therefore if V is perfect, the strong topology on V' is nothing more than the well known *compact–open topology* [5].

Theorem 4.15. *Let V be a perfect space with dual V'. Then a sequence $\{v'_k\}_{k=1}^{\infty}$ in V' converges strongly if and only if it converges weakly. (i.e. weak and strong converge coincide on the dual space V')*

Proof. Obviously strong convergence implies weak convergence. So take a sequence $\{v'_k\}_{k=1}^{\infty}$ in V' which converges weakly to $v' \in V'$. Without loss of generality we can take $v' = 0$ (replace v'_k with $v'_k - v'$). The sequence $\{v'_k\}_{k=1}^{\infty}$ is weakly bounded,

being weakly convergent. Thus by Theorem 4.10 we have that $\{v'_k\}_{k=1}^{\infty}$ is strongly bounded.

To show $\{v'_k\}_{k=1}^{\infty}$ converges strongly we must show $\langle v'_k, \cdot\rangle$ goes to 0 uniformly on each bounded set $D \subset V$. Suppose, by contradiction, that there exists a bounded set D in V where $\langle v'_k, \cdot\rangle$ does not go to 0 uniformly. So for some $\epsilon > 0$, there is a $k_1 \geq 1$ such that $\left|\langle v'_{k_1}, v\rangle\right| \geq \epsilon$ for some $v \in D$. Name this v as v_{k_1}. Likewise there an $k_2 > k_1$ and $v_{k_2} \in V$ such that $\left|\langle v'_{k_2}, v_{k_2}\rangle\right| \geq \epsilon$. Continuing in this manner we form a sequence $\{v_{k_j}\}_{j=1}^{\infty}$ in V. This sequence is bounded, being taken from the bounded set D.

Knowing that V is a perfect space we have a subsequence of $\{v_{k_j}\}_{j=1}^{\infty}$ that converges to some v in V. Renumbering if necessary we will just take this subsequence to be $\{v_{k_j}\}_{j=1}^{\infty}$. Since $\{v_{k_j}\}_{j=1}^{\infty}$ goes to v in V then the sequence given by $w_{k_j} = v_{k_j} - v$ goes to 0 in V.

Now for any strongly bounded set $B \subset V'$, Theorem 4.6 guarantees that $\langle v', w_{k_j}\rangle$ goes to 0 uniformly for all $v' \in B$. So take B to be the set $\{v'_{k_j}\}_{j=1}^{\infty}$. Then $\langle v'_{k_j}, w_{k_j}\rangle$ goes to 0 and by weak convergence we have that $\langle v_{k_j}, v\rangle$ goes to 0. Thus

$$\lim_{j\to\infty} \langle v'_{k_j}, v_{k_j}\rangle = \lim_{j\to\infty} \langle v'_{k_j}, w_{k_j}\rangle + \langle v'_{k_j}, v\rangle = 0.$$

This contradicts the construction of the v_{k_j} and v'_{k_j}. □

5. Inductive Limit Topology

Given a sequence of normed spaces $\{(W_n, |\cdot|_n); n \geq 1\}$ with W_n continuously imbedded in W_{n+1} for all n, we form the space $W = \bigcup_{n=1}^{\infty} W_n$ and endow W with the finest locally convex vector topology such that for each n the inclusion map $i_n : W_n \to W$ is continuous. This topology is call the *inductive limit topology* on W and W is said to be the *inductive limit* of the sequence $\{(W_n, |\cdot|_n); n \geq 1\}$

5.1. Local Base.

As always, when discussing a vector topology, we should try to discover what a useful local base for the topology would be.

Theorem 5.1. *Suppose W is the inductive limit of the normed spaces $\{(W_n, |\cdot|_n); n \geq 1\}$. A local base for W is given by the set $\mathcal{B}$ of all balanced convex subsets U of W such that $i_n^{-1}(U)$ is a neighborhood of 0 in W_n for all n.*

Proof. We first apply Lemma 1.4 to see the set $\mathcal{B}$ is in fact a local base for W. Take $U, V \in \mathcal{B}$, then clearly $U \cap V \in \mathcal{B}$. Now if $U \in \mathcal{B}$, then it is easy to see that $\alpha U \in \mathcal{B}$ for $\alpha \neq 0$. Finally we show $U \in \mathcal{B}$ is absorbing. Note that $i_n^{-1}(U)$ is absorbing in W_n (since W_n is a normed space and $i_n^{-1}(U)$ is open in W_n). Thus U absorbs all the point of $W_n = i_n(W_n) \subset W$. Since $W = \bigcup_{n=1}^{\infty} W_n$, U absorbs W. Thus by Lemma 1.4 we see that $\mathcal{B}$ is a base of neighborhoods for a locally convex vector topology on W.

It is fairly straightforward to see that $\mathcal{B}$ gives us the finest locally convex vector topology making all the $i_n : W_n \to W$ continuous: Let τ be a locally convex vector topology on W making all the i_n continuous. Take a convex neighborhood (of 0) U in τ. By Lemma 1.3 we can assume U is balanced. Since each i_n is continuous, we have $i_n^{-1}(U)$ is a neighborhood in W_n. Thus $U \in \mathcal{B}$. □

Corollary 5.2. *Suppose W is the inductive limit of the normed spaces $\{(W_n, |\cdot|_n); n \geq 1\}$. A local base for W is given by the balanced convex hulls of sets of the form $\bigcup_{n=1}^{\infty} i_n(B_n(\epsilon_n))$ (where $B_n(\epsilon_n) = \{x \in W_n \,;\, |x|_n < \epsilon_n\}$).*

Proof. Let U be the balanced convex hull of the set $\bigcup_{n=1}^{\infty} i_n(B_n(\epsilon_n))$ in W. Then $B_n(\epsilon_n) \subset i_n^{-1}(U)$. So $i_n^{-1}(U)$ is a neighborhood of 0 in W_n. By Theorem 5.1 such a U is a neighborhood in W.

Now if U is any balanced convex neighborhood of 0 in W, then $i_n^{-1}(U)$ contains a neighborhood $B_n(\epsilon_n)$. Hence $i_n(B_n(\epsilon_n)) \subset U$. Since U is convex and balanced, the balanced convex hull of $\bigcup_{n=1}^{\infty} i_n(B_n(\epsilon_n))$ is contained in U. Thus the sets described form a local base for W. □

5.2. **Inductive Limit Topology on V'.** Let V be a countably-normed space. Then V', the dual of V, can be regarded as the inductive limit of the sequence of normed spaces $\{(V_n', |\cdot|_{-n}); n \geq 1\}$. Thus V' can be given the inductive limit topology. In light of Proposition 3.3 and Proposition 4.2 we see that the inductive limit topology on V' is finer than the strong and weak topology on V'. We also have the following useful result about convergence on V' in the inductive topology:

Theorem 5.3. *Let V be a countably-normed space. Endow V' with the inductive limit topology. A sequence $\{v_k'\}_{k=1}^{\infty}$ converges to v' in V' if and only if there exists some n such that $v_k \in V_n'$ for all k and $\lim_{k\to\infty} |v_k' - v|_{-n} = 0$ (i.e. v_k' converges to v' in V_n').*

Proof. ($\Leftarrow$) Using Corollary 5.2, this direction is obvious.

($\Rightarrow$) Let $\{v_k'\}_{k=1}^{\infty}$ be a sequence in V' that converges to $v' \in V'$. Replacing v_k' with $v_k' - v'$, if necessary, we assume that $v' = 0$. Since $\{v_k'\}_{k=1}^{\infty}$ converges to 0 in the inductive limit topology, by the above discussion, it converges to 0 in the strong topology on V. Hence $\{v_k'\}_{k=1}^{\infty}$ is strongly bounded. Thus by Theorem 4.6 we have that there is an n such that $\{v_k'\}_{k=1}^{\infty} \subset V_n'$.

Now we must show that $|v_k'|_{-n}$ goes to 0 as k tends to infinity. That is for a given $\epsilon > 0$ we need to find a $K > 0$ such that for all $k \geq K$ we have $|v_k'|_{-n} < \epsilon$. Consider the base neighborhood U of V' given by the balanced convex hull of $\bigcup_{l=1}^{\infty} B_l$ where for $l = n$ we take

$$B_n = \{v' \in V_n' \,;\, |v'|_{-n} < \epsilon\}.$$

For $l < n$, $B_l = \{v' \in V_l' \,;\, |v'|_{-l} < \epsilon_l\}$ where $\epsilon_l > 0$ is chosen so that B_l is contained in $i_{l,n}'^{-1}(B_n)$. (Such an $\epsilon_l > 0$ exist by the continuity of the inclusion map $i_{l,n}' : V_l' \to V_n'$.)

And for $l > n$ we first note that the restricted inclusion $\tilde{i}_{n,l} : V_n' \to i_{n,l}(V_n') = V_n' \subset V_l'$ is a homeomorphism (since V_n' is continuously imbedded into V_{n+1}' for each n). This gives us $i_{n,l}'(B_n) \cap V_n' = W \cap V_n'$ where W is open in V_l'. Thus take $B_l = \{v' \in V_l' \,;\, |v'|_{-l} < \epsilon_l\}$ where $\epsilon_l > 0$ is chosen so that $B_l \subset W$.

Now since U is open, there is a K such that for all $k \geq K$ we have that $v_k' \in U$. We will show that $v_k' \in B_n$ for $k \geq K$. Let $k \geq K$ and consider the element v_k'. Since $v_k' \in U$ we can write v_k' as $v_k' = \sum_{j=1}^{m} l_j y_j$ where $\sum_{j=1}^{m} |l_j| \leq 1$ and $y_j \in B_j$. Observe that each y_j with $l_j \neq 0$ is in V_n'. (If there is an y_j not in V_n' with $l_j \neq 0$, then v_k' could not be V_n'.) Thus we have

$$|v_k'|_{-n} \leq \sum_{j=1}^{m} |l||y_j|_{-n}. \tag{5.1}$$

Observe that for $j \leq n$, $y_j \in B_j \subset i_{j,n}'^{-1}(B_n)$. So $|y_j|_{-n} < \epsilon$. Also for $j > n$ we have that $y_j \in B_j$. Since $y_j \in V_n'$ we get that $y_j \in B_j \cap V_n' \subset i_{n,j}'(B_n) \cap V_n'$. So $|y_j|_{-n} < \epsilon$.

Therefore in (5.1) we have that

$$|v_k'|_{-n} \le \sum_{j=1}^{\infty} |l||y_j|_{-n} < \sum_{j=1}^{\infty} |l_j|\epsilon \le \epsilon.$$

Thus v_k' is in B_n for all $k \ge K$ and we are done. □

6. Comparing the Three Topologies

In this section we compare the three topologies on the dual V' of a countably-normed space V. In order to do this efficiently we first introduce a fourth topology on V'. It is the *Mackey topology* on V'.

6.1. **Mackey Topology.** In order to talk about the Mackey topology we need the following notion:

Definition 6.1. Let $\mathcal{D}$ be a set of bounded subsets of a topological vector space E with dual E'. The topology of *uniform convergence on the sets of* $\mathcal{D}$ is the topology with subbasis neighborhoods of 0 given by

$$N(D;\epsilon) = \left\{v' \in E' \,;\, \sup_{v\in D} |\langle v', v\rangle| < \epsilon\right\},$$

where $D \in \mathcal{D}$ and $\epsilon > 0$. This is also referred to as the topology of *$\mathcal{D}$–convergence* on E'.

From the definition we see that a local base neighborhood for the topology of $\mathcal{D}$–convergence on a vector space E with dual E' looks like

$$N(D_1;\epsilon_1) \cap N(D_2;\epsilon_2) \cap \cdots N(D_k;\epsilon_k)\,,$$

where $D_j \in \mathcal{D}$ and $\epsilon_j > 0$ for all $1 \le j \le k$. We now state the following theorem without proof:

Theorem 6.2 (Mackey-Arens)**.** *Suppose that under a locally convex vector topology τ, E is a Hausdorff space. Then E has dual E' under τ if and only if τ is a topology of uniform convergence on a set of balanced convex weakly–compact subsets of E'.*

For a proof of this results see [6], [7], or [3]. Using this theorem we can define the Mackey topology as follows:

Definition 6.3. Let E be a topological vector space with dual E'. The *Mackey topology* on E is the topology on uniform convergence on all balanced convex weakly–compact subsets of E'.

Remark 6.4. From this discussion we see that the Mackey topology on V' has a local base given by

$$N(C;\epsilon) = \left\{v' \in V' \,;\, \sup_{v\in C} |\langle v', v\rangle| < \epsilon\right\},$$

where $\epsilon > 0$ and C is a balanced convex weakly–compact set in V.

Remark 6.5. Although we have not defined the term weakly–compact, it is nothing to fret about. Just as we have defined the weak topology on V', we can define an analogous topology on V. This topology has as its local base sets of the form

$$N(v_1', v_2', \ldots, v_k'; \epsilon) = \left\{v \in V;\ |\langle v_j', v\rangle| < \epsilon,\ 1 \le j \le k\right\}.$$

When a set in V is said to be weakly–compact, it simply means that the set is compact with respect to the weak topology on V.

6.2. **The Topologies on V'.** Let us make the following notational convention throughout this section:

Notation. Let V be a countably-normed space with dual V'. The weak topology, strong topology, inductive limit topology, and Mackey topology on V' with be denoted by τ_w, τ_s, τ_i, and τ_m, respectively.

Proposition 6.6. *Let V be a countably-normed space. Suppose $C \subset V$ is weakly–compact, then C is weakly bounded.*

Proof. Let $v' \in V'$ and $\epsilon > 0$ be given. We have to show there exists a k such that $C \subset kN(v'; \epsilon)$ (see Definition 2.12). Cover C by the sets $\{kN(v'; \epsilon)\}_{k=1}^{\infty}$. Since C is weakly–compact, $C \subset kN(v'; \epsilon)$ for some k. □

Corollary 6.7. *Let V be a countably-normed space with dual V'. Then the strong topology τ_s is finer than the Mackey topology τ_m on V'.*

Proof. The topology τ_s is by definition the topology of uniform convergence on all bounded sets in V. But by Theorem 2.13 every bounded set in V is weakly bounded. And by Proposition 6.6, we have that every weakly–compact set is weakly bounded. Thus $\tau_m \subset \tau_s$. □

Lemma 6.8. *Let V be a countably-normed space with dual V'. Then V' is Hausdorff in the weak topology t_w, and hence in the strong, Mackey, and inductive limit topologies.*

Proof. Take $u' \in V'$. We must find a neighborhood of 0 in τ_w that does not contain u'. Take $v \in V$ such that $|\langle u', v\rangle| \neq 0$. Let $|\langle u', v\rangle| = l \neq 0$. Consider the set $N(v; \frac{l}{2}) = \left\{v' \in V';\ |\langle v', v\rangle| < \frac{l}{2}\right\}$. This set cannot contain u'. Thus V' is Hausdorff in the weak topology (and hence in the finer strong, Mackey, and inductive topologies). □

Lemma 6.9. *Let V be a countably-Hilbert space with dual V'. Then the dual of V' is V when V' is given the weak, strong, Mackey, or inductive limit topology.*

Proof. Consider $v \in V$ and the corresponding linear functional $\hat{v}$ on V' given by

$$\langle \hat{v}, v'\rangle = \langle v', v\rangle \qquad \text{where } v' \in V'$$

Observe that $\langle \hat{v}, \cdot\rangle$ is continuous since $\langle \hat{v}, \cdot\rangle^{-1}(-\epsilon, \epsilon) = \{v' \in V'; |\langle v', v\rangle| < \epsilon\}$ which is open in the weak topology (and hence the strong, Mackey, and inductive limit topologies) on V'.

Also note that if $\hat{u} = \hat{v}$, then $\langle v', v\rangle = \langle v', u\rangle$ for all $v' \in V'$. Thus $v = u$. Therefore the correspondence $v \to \hat{v}$ is injective.

We now show that the correspondence $v \to \hat{v}$ is surjective. Take $v'' \in V''$, the dual of V'. Then v'' is continuous on V'. Since $V' = \bigcup_{n=1}^{\infty} V_n'$ by Proposition 2.9, we have that $v'' \in V_n''$ for all n. But $V_n = V_n''$ since V_n is a Hilbert space. Thus v'' can be considered as an element of V_n for all n. Since V is a complete we have that $\bigcap_{n=1}^{\infty} V_n = V$ by Proposition 2.4. Thus $v'' \in V$ and we have that $v \to \hat{v}$ is surjective. □

Theorem 6.10. *Let V be a countably-Hilbert space with dual V'. Then the inductive, strong, and Mackey topologies on V' are equivalent (i.e. $\tau_s = \tau_i = \tau_m$).*

Proof. By Lemma 6.8 and Lemma 6.9 we have that V' is Hausdorff and has dual V under the topologies t_s, t_i, and t_m. Thus we can apply Theorem 6.2 to V'. (In the theorem we are taking V' as E and V as E'.) This gives us that t_i and t_s are topologies of uniform convergence on a set of balanced convex weakly–compact subsets of V. However, by definition, the Mackey topology, τ_m, is the finest such topology. Thus $\tau_s \subset \tau_m$ and $\tau_i \subset \tau_m$. However, by Corollary 6.7 we have $\tau_m \subset \tau_s$. Thus $\tau_s = \tau_m$. Likewise, we have $\tau_i \subset \tau_m = \tau_s$; and, by Proposition 4.2 and the definition of the inductive limit topology on V' we have $\tau_s \subset \tau_i$. Therefore $\tau_s = \tau_m = \tau_i$. □

7. Borel Field

In this section our aim is to discuss the σ–field on V' generated by the three topologies (strong, weak, and inductive). We will see that under certain conditions the three σ–fields coincide. The standing assumption throughout this section is that V is a countably-Hilbert space with a countable dense subset Q_o. On each $V'_n \subset V'$ define the sets $F_n\left(\frac{1}{k}\right)$ for all k as:

$$F_n\left(\tfrac{1}{k}\right) = \left\{ v' \in V'_n \,;\, \sup_{v\in Q} |\langle v', \tfrac{v}{|v|_n}\rangle| < \tfrac{1}{k} \right\},$$

where $Q = Q_o - \{0\}$.

Recall that the local base for the topology of V'_n is given by the sets

$$N_n(\epsilon) = \{v' \in V'_n \,;\, |v'|_n < \epsilon\},$$

where $\epsilon > 0$.

Lemma 7.1. *In V'_n we have that $F_n\left(\frac{1}{k}\right) = N_n\left(\frac{1}{k}\right)$ for all k.*

Proof. Recall that $|v'|_{-n} = \sup_{v\in V_n - \{0\}} |\langle v', \frac{v}{|v|_n}\rangle|$. It is enough to show that for any $v' \in V'_n$ we have $|v'|_{-n} = \sup_{v\in Q} |\langle v', \frac{v}{|v|_n}\rangle|$. This is quite easy to see: for any non-zero $v \in V_n$ we have a sequence $\{v_l\}_{l=1}^{\infty}$ in Q that converges to v (since Q is dense in V and V is dense in V_n). Thus $\langle v', v\rangle = \lim_{l\to\infty}\langle v', v_l\rangle$. □

Proposition 7.2. *The collection $\{F_n(\frac{1}{k})\}_{k=1}^{\infty}$ forms a local base in V'_n. That is, V'_n is first countable.*

Proof. Take an open set $U \subset V'_n$ containing 0. Then $N_n(\epsilon) \subset U$ for some $\epsilon > 0$. Choose k so that $\frac{1}{k} < \epsilon$. Then by Lemma 7.1 we have $F_n\left(\frac{1}{k}\right) = N_n\left(\frac{1}{k}\right) \subset N_n(\epsilon)$. □

Since each V_n is a separable Hilbert space, so is its dual V'_n. Let Q'_n be a countable dense subset in V'_n.

Proposition 7.3. *The collection $\{x' + F_n\left(\frac{1}{k}\right) \;|\; x' \in Q'_n\,,\, 1 \le k < \infty\}$ is a basis for V'_n. That is, V'_n is second countable.*

Proof. Consider an open set $U \subset V'_n$ and an element v' in U. By Proposition 7.2 there is a k such that $v' + F_n\left(\frac{1}{k}\right) \subset U$. Take $x' \in Q'_n$ such that $|x' - v'|_{-n} < \frac{1}{2k}$.

Observe that $x' + F_n\left(\frac{1}{2k}\right) \subset v' + F_n\left(\frac{1}{k}\right)$: Take any $w' \in F_n\left(\frac{1}{2k}\right)$ and we have

$$\begin{aligned}\sup_{v\in Q} |\langle x' - v' + w', \tfrac{v}{|v|_n}\rangle| &\le |x' - v'|_{-n} + \sup_{v\in Q} |\langle w', \tfrac{v}{|v|_n}\rangle| \\ &< \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}.\end{aligned}$$

This gives us that $x' - v' + F_n\left(\frac{1}{2k}\right) \subset F_n\left(\frac{1}{k}\right)$ or equivalently $x' + F_n\left(\frac{1}{2k}\right) \subset v' + F_n\left(\frac{1}{k}\right)$. Also $v' \in x' + F_n\left(\frac{1}{2k}\right)$ since $|x' - v'|_{-n} < \frac{1}{2k}$.

In summary we have that $v' \in x' + F_n\big(\frac{1}{2k}\big) \subset v' + F_n\big(\frac{1}{k}\big) \subset U$. Therefore the collection $\{x' + F_n\big(\frac{1}{k}\big) \mid x' \in Q'_n\,, 1 \le k < \infty\}$ is a basis for V'_n □

Lemma 7.4. *Let $\sigma(\tau_w)$ be the Borel σ–field on V' induced by the weak topology. Then $F_n(\frac{1}{k})$ is in $\sigma(\tau_w)$ for all positive integers k and n.*

Proof. Observe $F_n\big(\frac{1}{k}\big) = \big\{v' \in V'_n\,;\, |v'|_n < \frac{1}{k}\big\} = \big\{v' \in V'\,;\, |v'|_{-n} < \frac{1}{k}\big\}$. (If $v' \in V'$ satisfies $\sup_{v\in Q} |\langle v', \frac{v}{|v|_n}\rangle| < \frac{1}{k}$, then $v' \in V'_n$.)

Now note that $F_n\big(\frac{1}{k}\big)$ can be expressed as

$$F_n\big(\tfrac{1}{k}\big) = \bigcup_{r\in S} \bigcap_{v\in Q'_n} N\big(\tfrac{v}{|v|_n}; r\big),$$

where $N\big(\frac{v}{|v|_n}; r\big) = \{v' \in V';\ |\langle v', \frac{v}{|v|_n}\rangle| < r\}$ and $S = \{r \in \mathbb{Q}\,;\ 0 < r < \frac{1}{k}\}$. Therefore $F_n\big(\frac{1}{k}\big)$ can be expressed as the countable intersection of the weakly open sets $N\big(\frac{v}{|v|_n}; r\big)$. Hence $F_n\big(\frac{1}{k}\big)$ is in $\sigma(\tau_w)$. □

Theorem 7.5. *Let V' be endowed with a topology τ. If τ is finer than τ_w and the inclusion map $i'_n : V'_n \to V'$ is continuous for all n, then the σ–fields generated by τ and τ_w are equal. (i.e. $\sigma(\tau_w) = \sigma(\tau)$)*

Proof. Let U be a set in τ. Then $U_n = i_n'^{-1}(U)$ is open in V'_n. By Proposition 7.3, U_n can be expressed as $U_n = \bigcup_{l\in T} x'_{n_l} + F_n(\frac{1}{k_l})$ where $x'_{n_l} \in Q'_n$ and T is countable. Then

$$\begin{aligned} U \cap V' &= U \cap \Big(\bigcup_{n=1}^{\infty} V'_n\Big) = \bigcup_{n=1}^{\infty} U \cap V'_n \\ &= \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \bigcup_{l\in T} x'_{n_l} + F_n\Big(\tfrac{1}{k_l}\Big)\,. \end{aligned}$$

Thus U can be expressed as a countable union of sets in $\sigma(\tau_w)$. Hence U is in $\sigma(\tau_w)$. Therefore $\sigma(\tau_w) = \sigma(\tau)$.

□

Corollary 7.6. *The σ–fields generated by the inductive, strong, and weak topologies on V' are equivalent. (i.e. $\sigma(\tau_w) = \sigma(\tau_s) = \sigma(\tau_i)$)*

Proof. We can apply Theorem 7.5 since i'_n is continuous with respect to τ_i and τ_s and also both τ_i and τ_s are finer than τ_w. □

The σ–field on V' generated by the weak, strong, or inductive topology is referred to as the *Borel field* on V'.

8. A Word on Nuclear Spaces

Let V be a countably-Hilbert space associated with an increasing sequence of inner-product norms $\{|\cdot|_n; n \ge 1\}$. Again let V_n be the completion of V with respect to the norm $|\cdot|_n$.

Definition 8.1. The countably-Hilbert space V is called a *nuclear space* if for any n, there exists $m \ge n$ such that the inclusion map from V_m into V_n is a Hilbert-Schmidt operator (i.e. there is an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for V_m such that $\sum_{k=1}^{\infty} |e_k|_n^2 < \infty$).

Remark 8.2. Note that a trace class operator is also a Hilbert-Schmidt operator and that the product of two Hilbert-Schmidt operators is a trace class operator. Thus V is a nuclear space if and only if for any n, there exists $m \geq n$ such that the inclusion map from V_m into V_n is a trace class operator.

Proposition 8.3. *Let V be a perfect space. Then V has a countable dense subset (i.e. V is separable).*

Proof. Recall that $V = \bigcap_{n=1}^{\infty} V_n$. We can divide this into two cases: either each V_n is separable or there exists a k such that V_k is not separable.

In the first scenario, since $V \subset V_1$ and V_1 is separable, we can find a countable set $Q_1 \subset V$ such that Q_1 is dense in V in the norm $|\cdot|_1$. Likewise, we can find $Q_2 \subset V$ that is dense in V with respect to the norm $|\cdot|_2$. Continuing in this manner, we form $Q_n \subset V$ for all $n \in \{1, 2, \ldots\}$. Let $Q = \bigcup_{n=1}^{\infty} Q_n$. We will now show Q is dense in V. Let $v \in V$. For each n we can find a $v_n \in Q_n$ such that $|v - v_n|_n < \frac{1}{n}$. Then for any $k < n$ we have that

$$|v - v_n|_k \leq |v - v_n|_n < \frac{1}{n}$$

Therefore, the sequence $\{v_n\}_{n=1}^{\infty}$ will converge to v in the space V.

In the second case, without loss of generality we can take V_1 to be nonseparable. Using the Axiom of Choice we can find an uncountable set S_1 in V of points bounded in the norm $|\cdot|_1$ with the distance between any two points being larger than a positive constant M. (That is, for $x, y \in S_1$, we have $|x - y|_1 \geq M$.) Likewise, since $V = \bigcup_{n=1}^{\infty} \{v \in V\,;\, |v|_2 \leq n\}$, there is an uncountable set $S_2 \subset S_1$, which is bounded in the norm $|\cdot|_2$. Continuing in this manner, for each n we form an uncountable set $S_n \subset S_{n-1}$ such that S_n is bounded in the norm $|\cdot|_n$. Note that for any $x, y \in S_n$, we have that

$$|x - y|_n \geq |x - y|_1 \geq M$$

From each S_k take an arbitrary point v_k and form the set $\{v_k\}_{k=1}^{\infty}$. Note that $\{v_k\}_{k=1}^{\infty}$ is bounded in V. However, by construction $\{v_k\}_{k=1}^{\infty}$ cannot contain a Cauchy sequence. Therefore, V cannot be perfect, a contradiction. □

Proposition 8.4. *If V is a nuclear space, then V is perfect.*

Proof. Let B be a bounded set in V. Denote the set B considered as a subset of V_n by B_n. Since B is bounded, each B_n is bounded in V_n. For $m < n$, let $i_{n,m} : V_n \to V_m$ be the inclusion map. Note that $i_{n,m}(B_n) = B_m$. Since V is a nuclear space the image of the bounded set B_n has compact closure in V_m. For $m = 1$, taking a sequence of elements $\{v_k\}_{k=1}^{\infty}$ in B, there is a subsequence $\{v_{k_1}\}_{k_1=1}^{\infty}$ that is Cauchy in the norm $|\cdot|_1$. Taking $m = 2$, we can find subsequence $\{v_{k_2}\}_{k_2=1}^{\infty}$ of $\{v_{k_1}\}_{k_1=1}^{\infty}$ that is Cauchy in the norm $|\cdot|_2$. Continuing in this way and forming the diagonal sequence $\{v_{k_j}\}_{j=1}^{\infty}$ we see that $\{v_{k_j}\}_{j=1}^{\infty}$ is Cauchy in every norm $|\cdot|_k$. Thus $\{v_{k_j}\}_{j=1}^{\infty}$ is Cauchy in V. Since V is complete, this sequence has a limit in V. Thus B is limit point compact. □

Combining the last two propositions, we see that all the results proved throughout this article apply to nuclear spaces. Most importantly, for a nuclear space, the strong and inductive topologies on the dual coincide and the σ–fields generated by the inductive, strong, and weak topologies are equal.

9. Gaussian Measure on the Dual of a Nuclear Space

Let E be a real separable Hilbert space with norm $|\cdot|_0$, and let A be a positive Hilbert-Schmidt operator on E. Thus E has an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of eigenvectors of A, with

$$Ae_n = \lambda_n e_n$$

and

$$\sum_{n\geq 0} |\lambda_n|^2 < \infty \text{ with each } \lambda_n > 0$$

Using the notation $W = \{0, 1, 2, ...\}$, we have the coordinate map

$$I : E \mapsto \mathbb{R}^W : f \mapsto \big(\langle f, e_n\rangle\big)_{n\in W}$$

Let

$$F_0 = I(E) = \Big\{(x_n)_{n\in W} : \sum_{n\in W} x_n^2 < \infty\Big\} \tag{9.1}$$

Now, for each $p \in W$, let

$$F_p = \{(x_n)_{n\in W} : \sum_{n\in W} \lambda_n^{-2p} x_n^2 < \infty\} \tag{9.2}$$

On F_p we have the inner-product $\langle\cdot,\cdot\rangle_p$ given by

$$\langle a, b\rangle_p = \sum_{n\in W} \lambda_n^{-2p} a_n b_n$$

This makes F_p a real Hilbert space, unitarily isomorphic to $L^2(W, \nu_p)$ where ν_p is the measure on W specified by $\nu_p(\{n\}) = \lambda_n^{-2p}$. Moreover, we have

$$F \stackrel{\text{def}}{=} \cap_{p\in W} F_p \subset \cdots F_2 \subset F_1 \subset F_0 = L^2(W, \nu_0) \tag{9.3}$$

and each inclusion $F_{p+1} \to F_p$ is Hilbert-Schmidt.

Now we pull this back to E. First set

$$\mathcal{E}_p = I^{-1}(F_p) = \{x \in E : \sum_{n\geq 0} \lambda_n^{-2p} |\langle x, e_n\rangle|^2 < \infty\} \tag{9.4}$$

It is readily checked that

$$\mathcal{E}_p = A^p(E) \tag{9.5}$$

On $\mathcal{E}_p$ we have the pull back inner-product $\langle\cdot,\cdot\rangle_p$, which works out to

$$\langle f, g\rangle_p = \langle A^{-p} f, A^{-p} g\rangle \tag{9.6}$$

Then we have the chain

$$\mathcal{E} \stackrel{\text{def}}{=} \cap_{p\in W} \mathcal{E}_p \subset \cdots \mathcal{E}_2 \subset \mathcal{E}_1 \subset E, \tag{9.7}$$

with each inclusion $\mathcal{E}_{p+1} \to \mathcal{E}_p$ being Hilbert-Schmidt.

Equip $\mathcal{E}$ with the topology generated by the norms $|\cdot|_p$. Then $\mathcal{E}$ is, by definition, a nuclear space. The vectors e_n all lie in $\mathcal{E}$ and the set of all rational-linear combinations of these vectors produces a countable dense subspace of $\mathcal{E}$. Since $\mathcal{E}$ is a nuclear space, the topological dual $\mathcal{E}'$ is the union of the duals $\mathcal{E}_p'$. In fact, we have:

$$\mathcal{E}' = \cup_{p\in W} \mathcal{E}_p' \supset \cdots \mathcal{E}_2' \supset \mathcal{E}_1' \supset E' \simeq E, \tag{9.8}$$

where in the last step we used the usual Hilbert space isomorphism between E and its dual E'.

Going over to the sequence space, $\mathcal{E}'_p$ corresponds to

$$F_{-p} \stackrel{\text{def}}{=} \{(x_n)_{n\in W} : \sum_{n\in W} \lambda_n^{2p} x_n^2 < \infty\} \tag{9.9}$$

The element $y \in F_{-p}$ corresponds to the linear functional on F_p given by

$$x \mapsto \sum_{n\in W} x_n y_n$$

which, by Cauchy-Schwartz, is well-defined and does define an element of the dual F'_p with norm equal to the square root of $\sum_{n\in W} \lambda_n^{2p} y_n^2 < \infty$.

Consider now the product space $\mathbb{R}^W$, along with the coordinate projection maps

$$\hat{X}_j : \mathbb{R}^W \to \mathbb{R} : x \mapsto x_j$$

for each $j \in W$. Equip $\mathbb{R}^W$ with the product σ–algebra, i.e. the smallest sigma-algebra with respect to which each projection map $\hat{X}_j$ is measurable. A fundamental result in probability measure theory (a special case of Kolmogorov's theorem, for instance) says that there is a unique probability measure ν on the product σ–algebra such that each function $\hat{X}_j$, viewed as a random variable, has standard Gaussian distribution. Thus,

$$\int_{\mathbb{R}^W} e^{it\hat{X}_j}\, d\nu = e^{-t^2/2}$$

for $t \in \mathbb{R}$, and every $j \in W$. The measure ν is the product of the standard Gaussian measure $e^{-x^2/2}(2\pi)^{-1/2}dx$ on each component $\mathbb{R}$ of the product space $\mathbb{R}^W$.

Since, for any $p \geq 1$, we have

$$\int_{\mathbb{R}^W} \sum_{j\in W} \lambda_j^{2p} x_j^2\, d\nu(x) = \sum_{j\in W} \lambda_j^{2p} < \infty,$$

it follows that

$$\nu(F_{-p}) = 1$$

for all $p \geq 1$. Thus $\nu(F') = 1$.

We can, therefore, transfer the measure ν back to $\mathcal{E}'$, obtaining a probability measure μ on the sigma–algebra of subsets of $\mathcal{E}'$ generated by the maps

$$\hat{e}_j : \mathcal{E}' \to \mathbb{R} : f \mapsto f(e_j),$$

where $\{e_j\}_{j\in W}$ is the orthonormal basis of E we started with (note that each e_j lies in $\mathcal{E} = \cap_{p\geq 0}\mathcal{E}_p$). This is clearly the sigma–algebra generated by the weak topology on $\mathcal{E}'$, which is equal to the sigma–algebras generated by the strong or inductive-limit topologies.

The above discussion gives a simple direct description of the measure μ. Its existence is also obtainable by applying the well–known Minlos theorem.

To summarize, are at the starting point of much of infinite–dimensional distribution theory (white noise analysis): Given a real, separable Hilbert space E and a positive Hilbert-Schmidt operator A on E, we have constructed a nuclear space $\mathcal{E}$ and a unique probability measure μ on the Borel sigma–algebra of the dual $\mathcal{E}'$ such that there is a linear map

$$E \to L^2(\mathcal{E}', \mu) : \xi \mapsto \hat{\xi},$$

satisfying

$$\int_{\mathcal{E}'} e^{it\hat{\xi}(x)}\, d\mu(x) = e^{-t^2|\xi|_0^2/2}$$

for every real t and $\xi \in E$. This measure μ is often called the (*standard*) *Gaussian measure* or the *white noise measure* and is the principal measure used white–noise analysis.

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